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Theories of Gravity in 2 + 1 Dimensions

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We discuss the failure of general relativity to provide a proper Newtonian limit when the spacetime dimensionality is reduced to 2 + 1 and try to bypass this difficulty by assuming alternative equations for the gravitational field. We investigate the properties of spacetimes generated by circularly symmetric matter distributions in two cases: weakening Einstein equations, and by considering the Brans–Dicke theory of gravity. A comparison with the corresponding Newtonian picture is made.

1. INTRODUCTION

The present attention theoretical physicists devote to lower-dimensional gravity has brought to light the unsolved problem concerning the nonexistence of the Newtonian limit of general relativity when the spacetime dimensionality d is less than four (Jackiw, 1985; Deser *et al.*, 1984). This results basically from the fact that when $d = 2 + 1$ the Riemannian curvature is completely determined by the Einstein tensor ($R_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}G^{\rho\gamma}$). For $d = 1 + 1$ the situation is more drastic, since in this dimensionality $G_{\mu\nu}$ vanishes identically. As a consequence in the first case spacetime must be flat in regions where matter is absent. In the second case, as was pointed out by Jackiw (1985), "gravity has to be invented anew since general relativity cannot even be formulated".

In particular, if matter creates no gravitational field outside its location, neither 'planetary' motions nor gravitational waves are allowed to exist in a 2 + 1 spacetime.

In this paper, we restrict ourselves to a (2 + 1)-manifold and examine what happens to the above situation when the Einstein field equations are modified. Thus, we take up the classical problem of determining the

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gravitational field generated by a spherically (or, better, circularly) symmetric distribution of matter. We approach this problem in two different ways. First, we 'weaken' the Einstein equations in much the same way as did Jackiw (1985) in his attempt to formulate gravity in $1+1$ dimensions. Second, we consider the same problem in the light of the Brans–Dicke theory of gravity.

2. NEWTON'S THEORY OF GRAVITY IN 2 + 1 DIMENSIONS

It is generally accepted that a Newtonian gravitational field due to a spherical matter distribution of mass M and radius α in a $d=(n+1)$-dimensional spacetime should be expressed by the generalized law (see, for instance, Elmer and Olenick, 1982; Wilkins, 1984)

$$g(r) = -GM/r^{n-1} \tag{1}$$

where G is a constant and r is the distance from the center of the distribution, with $r > \alpha$. Thus, if $n = 2$, this corresponds to the gravitational potential

$$V(r) = GM \ln r \tag{2}$$

Then, the equations of motion for a test particle of mass m put in a region exterior to the matter distribution would be given by

$$mr^2\dot{\theta} = \text{const} = L \tag{3}$$

$$m\ddot{r} = L^2/mr^3 - GmM/r \tag{4}$$

where r and θ are polar coordinates and L is the angular momentum of the particle. On the other hand, the energy conservation equations yield directly

$$m\dot{r}^2/2 = E - (1/2m)L^2/r^2 - mMG \ln r \tag{5}$$

with E being the total energy of the particle.

It is clear from the latter equations that the particle cannot escape from the center of force, the permissible orbits being bounded. An illustrative picture may be obtained if we display these orbits in the particle's phase space, where p_r is the radial component of the momentum (see Figure 1). In this diagram the equilibrium point r_0 [which has the topology of a center (Andronov *et al.*, 1973)] represents the circular orbit $r = r_0 = Lm^{-1}(MG)^{-1/2}$, corresponding to the energy $E_0 = (1/2m)L/r_0^2 + mMG \ln r_0$ and a period $\tau = 2\pi L(mMG)^{-1}$. So, we arrive at the conclusion that in a

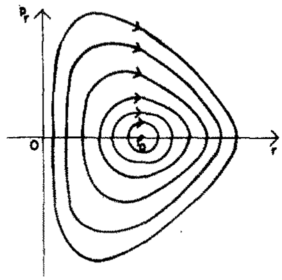

Fig. 1. Phase space trajectories of a particle subjected to a gravitational field in a (2 + 1)-dimensional Newtonian universe.

Newtonian universe with 2 + 1 dimensions no matter how large its energy is, a test particle is constrained to move within a bounded region of space.

3. EINSTEIN'S THEORY OF GRAVITY IN 2 + 1 DIMENSIONS

To find the motion of a test particle under the influence of the gravitational field generated by a matter distribution in any metric theory of gravity reduces to the problem of finding the spacetime geodesics. Thus, one has to know the geometry of that spacetime, which, in turn, must be determined from the gravitational field equations.

As mentioned earlier, if one considers Einstein's theory of gravity in 2 + 1 dimensions one is led to the conclusion that a test particle does not 'feel' the gravitational field in regions where the matter is absent. The spacetime is flat ($R_{\mu\nu\alpha\beta} = 0$) and the geodesics are simple straight lines. Thus, the situation here seems to differ drastically from the Newtonian picture, specifically if we regard the previously analyzed problem of the motion of particles under the influence of circularly symmetric massive bodies. And, since the curvature is null everywhere except in the interior of the matter distribution, there is no way to obtain a Newtonian limit.

Recently, there has been great interest in metric configurations exhibiting *topological defects* (Deser *et al.*, 1984). Essentially, these refer to the properties of a locally flat spacetime which nevertheless presents global features allowing one to distinguish it from a pure Minkowskian geometrical structure. The main quoted example of this phenomenon is the spacetime generated by an infinite static matter string in 3 + 1 dimensions which is described by a Riemannian flat geometry with bidimensional spatial conic sections (Vilenkin, 1981; Hiscock, 1985). The (2 + 1)-dimensional analog of this configuration may be generated by any circularly symmetric matter distribution. Since the geodesics in both cases are not simply straight lines in a Minkowskian spacetime, particles moving in these conic geometries are said 'to detect' the gravitational field in a number of effects whenever global variables (which involves integration along a closed contour) are measured (Bezerra, 1990). However, as the trajectories of test particles in these spacetimes are not bounded, 'planetary' motions not being allowed, there is no possibility of a Newtonian limit to exist.

4. JACKIW'S SCALAR EQUATION FOR GRAVITY IN 2 + 1 DIMENSIONS

As remarked before, the Einstein's tensor $G_{\mu\nu}$ vanishes identically in a $(1+1)$-spacetime manifold. An attempt to formulate the field equations in this dimensionality was put forward by Jackiw (1985). In this approach Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)R = T_{\mu\nu} \tag{6}$$

are 'weakened' by replacing (6) by its trace. In this way, we are left with the scalar equation

$$R = T \tag{7}$$

where $T = T^{\mu}_{\mu}$ is the trace of the energy-momentum tensor.

We shall assume (7) as a plausible field equation describing gravity also in a $2+1$ manifold. Now, considering a static, circularly symmetrical matter distribution as the source of the curvature, the metric coefficients should be a function of the radial coordinate r only. Space $T = 0$ in regions where matter is absent, the equation

$$R = 0 \tag{8}$$

reduces to a second ordinary differential equation involving the metric functions in the variable r. On the other hand, the most general form of a static, circularly symmetric field may be given by the line element

$$ds^2 = e^{2N}\, dt^2 - e^{2P}\, dr^2 - r^2\, d\theta^2 \tag{9}$$

where N and P are functions of the radial coordinate r only (see, for example, Cornish and Frankel, 1991). However, if N and P are independent, then they cannot be determined by equation (8) alone. A way to bypass this difficulty is to reduce the number of degrees of freedom of the geometry by choosing a metric tensor with only one degree of freedom. A natural choice is to consider a static, circularly conformally flat metric given by

$$ds^2 = f(r)(dt^2 - dr^2 - r^2\, d\theta^2) \tag{10}$$

which, as we shall see later, has the property of leading to the correct (2 + 1)-Newtonian limit in the weak-field approximation. Putting (10) into (8), we get the equation

$$f'' - \left(\frac{3f'^2}{4f}\right) + \frac{f'}{r} = 0 \tag{11}$$

whose solution is given by

$$f(r) = B\left(\ln\left(\frac{r}{A}\right)\right)^4 \tag{12}$$

where $A \geq 0$ and $B \geq 0$ are constants. Thus, the conformally flat solution of Jackiw's vacuum equation is given by the line element

$$ds^2 = B\left(\ln\left(\frac{r}{A}\right)\right)^4 (dt^2 - dr^2 - r^2\, d\theta^2) \tag{13}$$

This solution has a singularity, as may be readily seen by evaluating the value assumed by the invariant $R_{\mu\nu}R^{\mu\nu}$ at the surface $r = A$. However, as was pointed out by Cornish and Frankel (1991) (who found a similar solution in the weak-field approximation of equation (8)], this surface does not represent an event horizon, since light is not affected by the gravitational field nor does a change in the metric signature take place. On the other hand, it is worthwhile to mention that as far as behavior at infinity is concerned this geometry presents no asymptotic flatness, and this is a question deserving a further comment.

Let us investigate the motion of a massive test particle in this geometry. We begin by writing down the geodesic equations:

$$f\, dt/ds = \alpha \tag{14a}$$

$$fr^2\, d\theta/ds = l \tag{14b}$$

where α and l are integration constants. Now, if we divide (13) by ds^2 and use (14) we get the following first integral:

$$\frac{\mu\dot{r}^2}{2} + \frac{l^2}{r^2} + f(r) = \frac{\mu}{2} \tag{15}$$

in which we have put $\alpha^2 = \mu/2$ and dot means derivative with respect to the time coordinate. Now, this equation may be regarded as the analog of (5), i.e., the law of energy conservation in Newtonian gravity, if we formally define a gravitational *potential energy* given by $V = f(r) = B[\ln(r/A)]^4$. Then, we have an 'effective' potential $V_{\text{eff}} = l^2/r^2 + B[\ln(r/A)]^4$ which determines the radial motion of the particle. A simple analysis of the form of V_{eff} (see Figure 2) show us that the motion of particles in this spacetime is bounded.

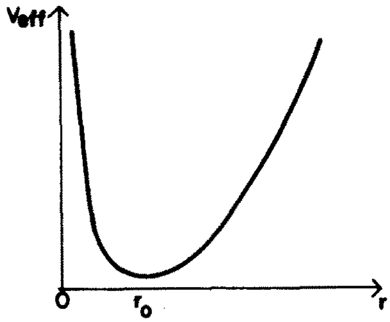

Fig. 2. Effective potential determining the radial motion of a particle in $(2+1)$-dimensional Jackiw gravity.

As in the case of Newtonian gravity, the equation $r = r_0$, with r_0 corresponding to the minimum of $V_{\text{eff}}(r)$, characterizes a circular motion of the particle around the center of the matter distribution.

At this point, it is interesting to note that a particle with no angular momentum ($l = 0$) keeps oscillating around r_0, which means that in addition to the usual repulsion force represented by the centrifugal term l^2/r^2, the particle feels a kind of 'gravitational repulsion force' during parts of its motion. Such effect has no Newtonian analogy. However, apart from this, we conclude that in $2+1$ dimensions the motion of particles in the spacetime of equation (13), which represents a circularly symmetric solution of Jackiw's gravity, and the motion of particles in Newtonian gravity exhibit a rather similar physical picture.

Finally, we should point out that Jackiw's scalar equation (7) leads to the Newtonian limit for the metric (10) if we use the same argument due to Cornish and Frankel (1991) for a general conformally flat metric $g_{\mu\nu} = f n_{\mu\nu}$ in $2+1$ dimensions.

5. BRANS–DICKE THEORY OF GRAVITY IN 2+1 DIMENSIONS

In this section let us consider the Brans–Dicke theory of gravity in $2+1$ dimensions and apply it to solve the same problem of obtaining the exterior gravitational field due to a circularly symmetric matter distribution.

The Brans–Dicke field equations in the absence of matter are given by

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = +(\omega/\phi^2)(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\phi^{,\beta}\phi_{,\beta}) + (1/\phi)(\phi_{,\mu;\nu}) \quad (16a)$$

$$\Box\phi = 0 \quad (16b)$$

where ϕ is the scalar field and ω is a free parameter to be determined by experiments. In $3+1$ spacetime, usually (but not always) the theory is expected to reduce to general relativity when $\omega \to \infty$ (Romero and Barros, 1993).

Since we are assuming a static and circularly symmetric matter distribution, we should start from the metric tensor given by equation (9) and a scalar field $\phi = \phi(r)$. Then, equations (16a) and (16b) become

$$P'/r = \omega\psi^2/2 - N'\psi \tag{17a}$$

$$N'/r = (\omega/2 + 1)\psi^2 - P'\psi + \psi' \tag{17b}$$

$$N'^2 - N'P' + N'' = -\omega\psi^2/2 + \psi/r \tag{17c}$$

where $\psi = \phi'/\phi$ and $\phi' = d\phi/dr$, $P' = dP/dr$, etc. The general solution of this system of equations leads to the metric (after some obvious simplifying coordinate transformations)

$$ds^2 = r^{2D}\, dt^2 - \lambda r^{2(D+B)}\, dr^2 - r^2\, d\theta^2 \tag{18a}$$

D, B, and λ are integration constants with $D = B(B+1)^{-1}(BW/2 - 1)$. On the other hand, the scalar field is given by

$$\phi = \phi_0 r^B \tag{18b}$$

with $\phi_0 = \text{const}$.

Looking at equation (18a), we verify that this metric has the following properties: (a) it has no singularities for $r \neq 0$; (b) the spacetime is not asymptotically flat.

6. FINAL REMARKS

The investigation of gravitation in $2+1$ dimensions was primarily concerned with the failure to construct a successful quantum theory of gravity in $3+1$ dimensions. Nevertheless, the subject has recently called the attention of theorists to some of its nonquantum aspects, such as the problem of the 'breakdown' of general relativity in lower dimensions. At this point, it seems that a natural and legitimate question arises inevitably: what theory could substitute for general relativity in lower dimensions? Should such a theory, at least from an epistemological point of view, have what is usually called a 'Newtonian limit'?

APPENDIX

The constant A in equation (13) can be eliminated by the coordinate transformation $dt = A\, d\tau$, $dr = A\, d\rho$. Thus, puttng $C = A^2 B$, the line element takes the simpler form

$$ds^2 = C(\ln \rho)^4(d\tau^2 - d\rho^2 - \rho^2\, d\theta^2) \tag{A1}$$

Going further, it is possible to determine the remaining constant which appears in (A1) in terms of the mass M of the matter distribution. Indeed, let us consider the following procedure. First we make the coordinate transformation $\rho = e^{1/K}r$, $\tau = e^{1/K}t$ and redefine the constant C by putting $C = e^{-2/K}K^4$. Thus the line element (A1) becomes

$$ds^2 = \lambda(r)(dt^2 - dr^2 - r^2\, d\theta^2) \tag{A2}$$

where $\lambda(r) = [\ln(er^K)]^4$. Now, the constant K must be a function of M and we can see that if $K \to 0$ then $\lambda(r) \to 1$ and (A2) goes over the metric of Minkowski spacetime which corresponds to $M = 0$. Hence, for small values of K we have $g_{00} \simeq 1 + 4K \ln(r)$. On the other hand, in the weak-field approximation we have $g_{00} \simeq 1 + 2\varphi$ (see ref. [10]), where we are taking the speed of light $c = 1$ and φ is given by eq. (2). Comparing the two expressions for g_{00} we get $K = GM/2$.

Analogously, in the case of Brans–Dicke metric (eq. 18a) the constant B is determined by the simple following argument. As is well known, if there is no matter ($M = 0$) the spacetime must be flat and Brans–Dicke theory should give the same result as Einstein theory. In this case the scalar field ϕ is constant and is to be identified to G^{-1}. As a consequence, B and D tend to zero if $M \to 0$. Again, in the weak-field approximation we must have $g_{00} \simeq 1 + D \ln(r)$. Applying the same reasoning as before we conclude that $D = 2GM$ and $\phi_0 = G^{-1}$.

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